Sequences and Series

4

LEARNING OBJECTIVES

In this chapter, you learn how to use GC to

- Generate a sequence using seq ( ).
- Find the Sum of a Sequence using sum ( ) or cumSum ( ).
- Find the Sum to Infinity of a GP.
- Find the nth term of a Recurring Sequence using SEQ mode.
- Solve Quadratic Equation using PlySmlt2 application.
- Evaluate $\sum$.
- Find the Root(s) of an Equation under GRAPH mode.
- Determine the Behaviour of a Sequence.

Example 4.1

A geometric series has common ratio r, and an arithmetic series has first term a and common difference d, where a and d are non-zero. The first three terms of the geometric series are equal to the first, sixth and tenth terms respectively of the arithmetic series.

(i) Show that $5r^2 - 9r + 4 = 0$.

(ii) Deduce that the geometric series is convergent and find, in terms of a, the sum to infinity.

(iii) The sum of the first n terms of the geometric series is denoted by S. Given that $a > 0$, find the least value of n for which S exceeds 99% of the sum to infinity.

Solution

(i) $r = \dfrac{a+(6-1)d}{a} = \dfrac{a+(10-1)d}{a+(6-1)d}$

$r = 1+5\left(\dfrac{d}{a}\right) = \dfrac{1+9\left(\dfrac{d}{a}\right)}{1+5\left(\dfrac{d}{a}\right)}$

If x, y and z are in G.P.,

$\dfrac{y}{x} = \dfrac{z}{y} = r$

$\therefore \dfrac{d}{a} = \dfrac{r-1}{5} \Rightarrow r = \dfrac{1+9\left(\dfrac{r-1}{5}\right)}{1+5\left(\dfrac{r-1}{5}\right)} \Rightarrow r^2 = 1+9\left(\dfrac{r-1}{5}\right) \Rightarrow 5r^2 = 5+9r-9$

$\Rightarrow 5r^2 - 9r + 4 = 0$ (shown)

(ii) $5r^2 - 9r + 4 = 0 \;\Rightarrow (r-1)(5r-4) = 0 \Rightarrow r = 1$ or $r = \dfrac{4}{5}$

Since $d \neq 0$, the three terms of GP are not the same, thus $r \neq 1$.

Hence, $r = \dfrac{4}{5} \Rightarrow$ Since $|r| < 1$, the geometric series is convergent.

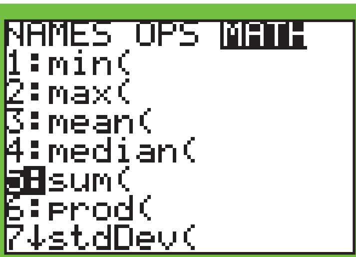
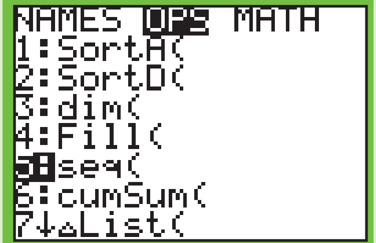
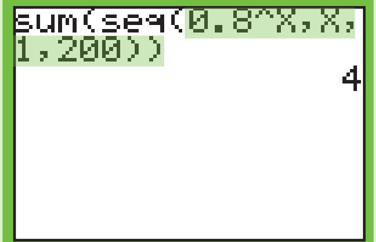
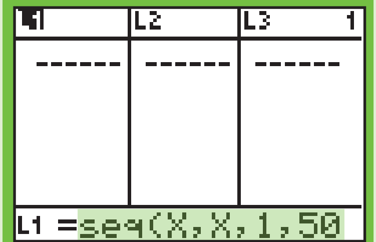
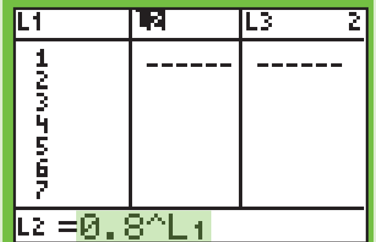
And the sum to infinity $= \dfrac{a}{1-\dfrac{4}{5}} = 5a$.

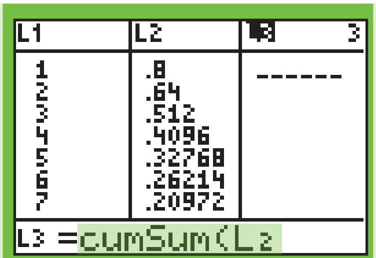
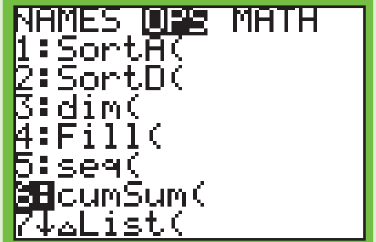
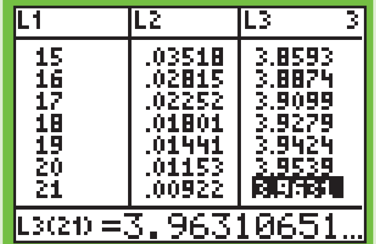
(iii)

$$S_n > 0.99 S_\infty$$
$$\frac{a(1-0.8^n)}{1-0.8} > 0.99\frac{a}{1-0.8}$$
$$1-0.8^n > 0.99$$
$$0.8^n < 0.01$$
$$n\lg 0.8 < \lg 0.01$$
$$n > \frac{\lg 0.01}{\lg 0.8}$$
$$n > 20.6$$

Hence, the least value of n is 21.

(You may use GC to verify your answer.)

2nd STAT ► ► 5		Select **LIST**, **MATH**, **sum(**. Note that **sum (LIST)**.
2nd STAT ► ► 5		Select **LIST**, **OPS**, **seq(**. Note that **seq (expression, variable, begin value, end value)**.
[Complete the rest of the expression as shown.]		Answer: "4" which is where $a = 0.8$ and $r = 0.8$. (Here, we assume a takes a positive value.)
STAT 1 ▲ [Type in the expression as shown.]		Select **STAT**, **EDIT**, **Edit**. Go up to the title L1. Generate a list of integers from 1 to 50 under L1.
► ▲ [Type in the expression as shown.]		Go up to the title L2. Generate a list of values of 0.8^{L1} under L2.

[▶][▲] [Type in the expression as shown.]		Go up to the title L3. Generate a list of cumulative values of L2 under L3.
		For cumSum, press [2nd][STAT][▶][6], i.e. **LIST**, **OPS**, **cumSum(**. Note that the syntax of cumSum is **cumSum (LIST)**.
[▶] … until a value > 4 × 0.99, i.e. 3.96, is reached.		∴ We obtain the least value of n is 21. ☑

Example 4.2

(i) Patrick saves \$20 on 1 January 2008. On the first day of each subsequent month he saves \$4 more than in the previous month, so that he saves \$24 on 1 February 2008, \$28 on 1 March 2009, and so on. On what date will he first have saved over \$5000 in total?

(ii) Kenny puts \$20 on 1 January 2008 into a bank account which pays compound interest at a rate of 3% per month on the last day of each month. He puts a further \$20 into the account on the first day of each subsequent month.

(a) How much compound interest has his original \$20 earned at the end of 3 years?

(b) How much in total, correct to the nearest dollar, is in the account at the end of 3 years?

(c) After how many complete months will the total in the account first exceed \$5000?

Solution

(i) $T_1 = 20$, $d = 4$

$$\frac{n}{2}\left[2(20)+4(n-1)\right] > 5000 \Rightarrow n^2 + 9n - 2500 > 0 \Rightarrow n < -54.7 \text{ or } n > 45.7$$

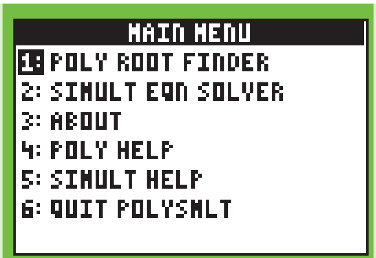
∴ Hence, Patrick will first have saved over \$5000 in total on 1 October 2011.

(You may use GC to find the roots.)

APPS 5

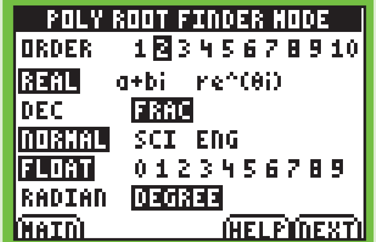
Select **APPLICATIONS, PlySmlt2.**

[1]



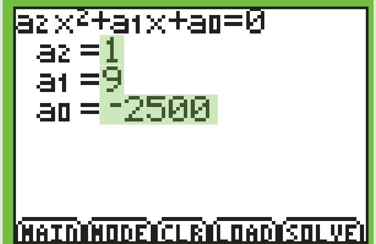
Select **POLY ROOT FINDER.**

[GRAPH]

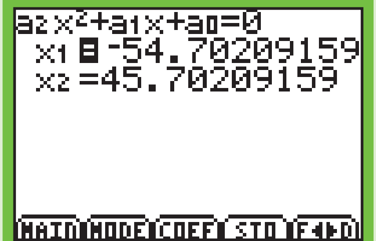


Select **NEXT.**

[Enter the coefficients accordingly.]



[GRAPH]



Select **SOLVE.**

Answers: 45.7 or −54.7

(ii) (a) The required compound interest $= 20(1.03)^{36} - 20 = \$38.0$ (3 s.f.)

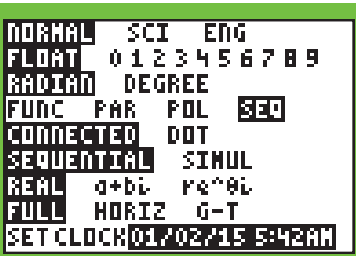
(b) The required amount $= 20(1.03) + 20(1.03)^2 + \ldots + 20(1.03)^{36}$

$$= \frac{20(1.03)\left[(1.03)^{36} - 1\right]}{1.03 - 1}$$

$$= \$1303$$

(You may use GC to verify your answer.)

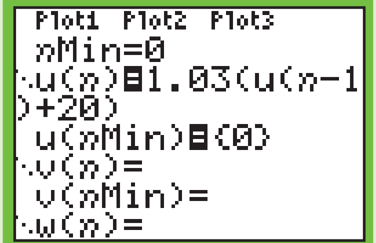
[MODE] [▼][▼][▼][▶][▶][▶] [ENTER]



Enter **MODE** settings.

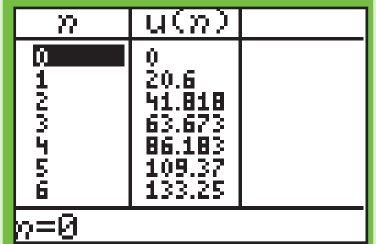
Select **SEQ**.

[Enter the data as shown.]



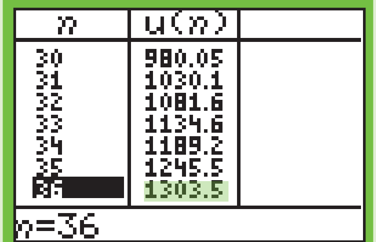
Press [2nd][7] for **u** and [X,T,Θ,*n*] for **n**.

[2nd][GRAPH]



Select **TABLE.**

[▶] … until **n = 36** is reached.



∴ u(36) = 1303.5. ☑

(c) $20(1.03)+20(1.03)^2+...+20(1.03)^n > 5000$

$$\frac{20(1.03)(1.03^n-1)}{1.03-1} > 5000$$

$$1.03^n > 8.282$$

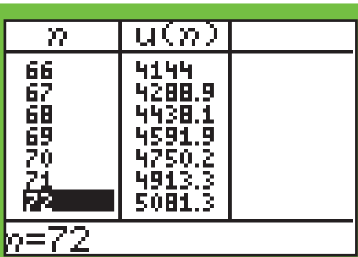
$$n > \frac{\lg 8.282}{\lg 1.03}$$

$$n > 71.5$$

The amount will first exceed \$5000 after 72 months.

(You may use GC to verify your answer.)

▶ … until u(n) first reaches 5000 or above.



Note that
u(71) = 4913.3 < 5000
u(72) = 5081.3 > 5000
∴ $n = 72$. ☑

Example 4.3

A sequence u_1, u_2, u_3, … is such that $u_1 = 1$ and

$$u_{n+1} = u_n - \frac{3n^2+3n+1}{n^3(n+1)^3}, \text{ for all } n \ge 1.$$

(i) Use the method of mathematical induction to prove that $u_n = \dfrac{1}{n^3}$.

(ii) Hence find $\displaystyle\sum_{n=1}^{N} \frac{3n^2+3n+1}{n^3(n+1)^3}$.

(iii) Give a reason why the series in part (ii) is convergent and state the sum to infinity.

(iv) Use your answer to part (ii) to find $\displaystyle\sum_{n=2}^{N} \frac{3n^2-3n+1}{n^3(n-1)^3}$.

Solution

(i) Let P_n be the statement $u_n = \dfrac{1}{n^3}$ where $u_{n+1} = u_n - \dfrac{3n^2+3n+1}{n^3(n+1)^3}$ and $u_1 = 1$ for $n \in \mathbb{Z}^+$.

When $n = 1$, L.H.S. $= u_1 = 1$; R.H.S. $= \dfrac{1}{1^2} = 1$.

$\because$ L.H.S. = R.H.S.

$\therefore$ P_1 is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = \dfrac{1}{k^3}$.

To prove P_{k+1} is true, i.e. $u_{k+1} = \dfrac{1}{(k+1)^3}$

$$\begin{aligned}
\text{L.H.S.} = u_{k+1} &= u_k - \frac{3k^2+3k+1}{k^3(k+1)^3} \\
&= \frac{1}{k^3} - \frac{3k^2+3k+1}{k^3(k+1)^3} \\
&= \frac{(k+1)^3 - 3k^2 - 3k - 1}{k^3(k+1)^3} \\
&= \frac{k^3 + 3k^2 + 3k + 1 - 3k^2 - 3k - 1}{k^3(k+1)^3} \\
&= \frac{1}{(k+1)^3} \\
&= \text{R.H.S.}
\end{aligned}$$

$\therefore$ P_{k+1} is true whenever P_k is true.

Since P_1 is true and $P_k \Rightarrow P_{k+1}$ is true by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

(ii) $$\sum_{n=1}^{N} \frac{3n^2+3n+1}{n^3(n+1)^3} = \sum_{n=1}^{N} (u_n - u_{n+1})$$

$$\begin{aligned}
= \; & u_1 - \cancel{u_2} \\
& + \cancel{u_2} - \cancel{u_3} \\
& \dots \\
& + \cancel{u_{N-1}} - \cancel{u_N} \\
& + \cancel{u_N} - u_{N+1} \\
= \; & u_1 - u_{N+1} \\
= \; & 1 - \frac{1}{(N+1)^3}
\end{aligned}$$

(You may use GC to verify your answer.)

STAT 1

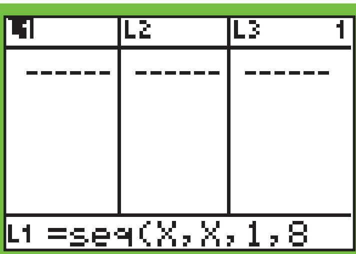
Select **STAT**, **EDIT**, **Edit**.

▲

Go up to the title L1.

[Type in the expression as shown.]

Generate a list of integers from 1 to 8 under L1.

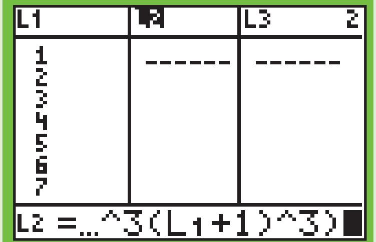


▶ ▲

Go up to the title L2.

Type in L2 $= \frac{3L_1^{\ 2}+3L_1+1}{L_1^{\ 3}(L1+L)^3}$.

Generate a list of values based on the given formula under L2.

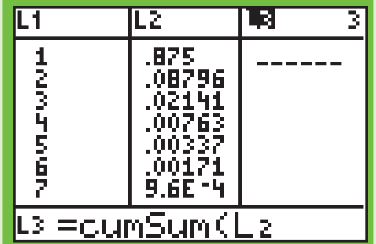


▶ ▲

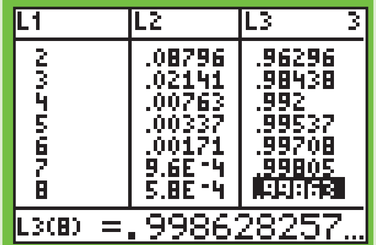
Go up to the title L3.

[Type in the expression as shown.]

Generate a list of cumulative values of L2 under L3.

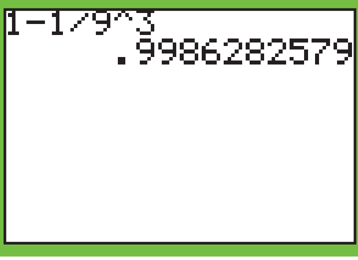


▶ … until **L1=8** is reached.



2nd MODE

Evaluate the answer in part (ii) when $N = 8$.



Note that the 2 values are the same. Hence, the expression obtained in part (ii) is verified. ☑

UNIT 4 SEQUENCES AND SERIES

(iii) When $N \to \infty, \dfrac{1}{(N+1)^3} \to 0 \Rightarrow 1 - \dfrac{1}{(N+1)^3} \to 1 \Rightarrow \displaystyle\sum_{n=1}^{N} \frac{3n^2+3n+1}{n^3(n+1)^3} \to 1$

Hence, the series in part (ii) is convergent and the sum to infinity is 1.

(iv) $$\sum_{n=2}^{N} \frac{3n^2-3n+1}{n^3(n-1)^3} = \sum_{r=1}^{N-1} \frac{3(r+1)^2-3(r+1)+1}{(r+1)^3 r^3} \text{ when } n = r+1$$

$$= \sum_{r=1}^{N-1} \frac{3(r+1)^2-3(r+1)+1}{r^3(r+1)^3}$$

$$= \sum_{r=1}^{N-1} \frac{3r^2+6r+3-3r-3+1}{r^3(r+1)^3}$$

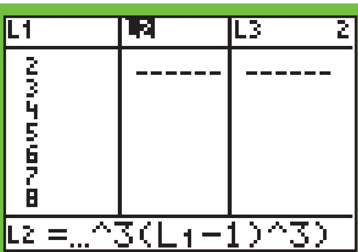
$$= \sum_{r=1}^{N-1} \frac{3r^2+3r+1}{r^3(r+1)^3}$$

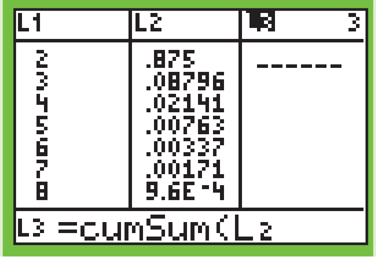
$$= 1 - \frac{1}{(N-1+1)^3}$$

$$= 1 - \frac{1}{N^3}$$

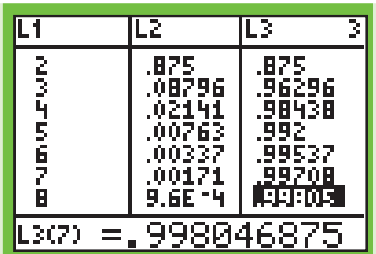
(You may use GC again to verify your answer.)

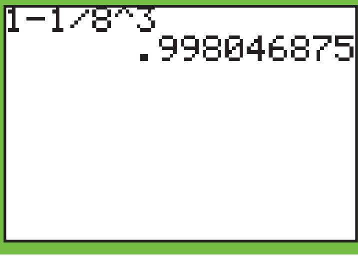
[Apply similar keystrokes shown in part (ii).]





Note that L1 now starts from 2.

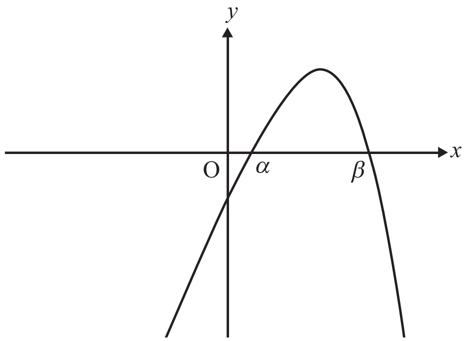




Evaluate the answer in part (ii) when $N = 8$.

Note that the 2 values are the same. Hence, the expression obtained in part (ii) is verified. ☑

Example 4.4



The diagram shows the graph of $y = 2x - e^{\frac{x}{2}}$. The two roots of the equation are denoted by α and β. where $\alpha < \beta$.

(i) Find the values of α and β, each correct to 3 decimal places.

A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \frac{1}{2}e^{\frac{x_n}{2}}$$

for $n \geq 1$.

(ii) Prove algebraically that, if the sequence converges, then it converges to either α or β.

(iii) Use a calculator to determine the behaviours of the sequence for each of the cases $x_1 = 0$, $x_1 = 3$, $x_1 = 6$.

(iv) By considering $x_{n+1} - x_n$, prove that

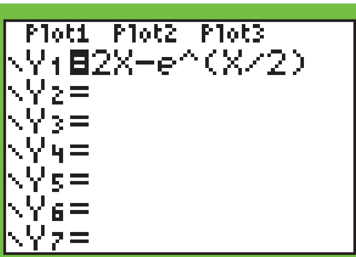
$$x_{n+1} < x_n \text{ if } \alpha < x_n < \beta,$$
$$x_{n+1} > x_n \text{ if } x_n < \alpha \text{ or } x_n > \beta$$

(v) State briefly how the results in part (iv) relate to the behaviours determined in part (iii).

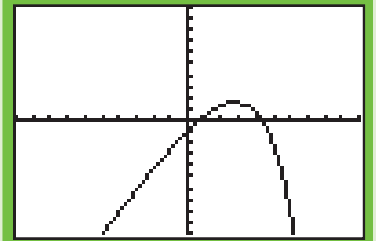
Solution

[Y=]

[Enter the equation as shown.]

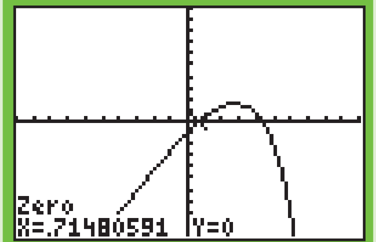


[GRAPH]



[2nd][TRACE]
[2]

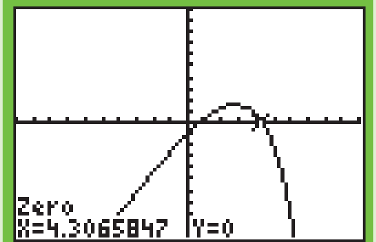
[Choose the point as shown in the graph.]



Select CALC.
Select zero.

$\therefore\ \alpha = 0.715.$

[Apply similar steps for the 2nd root.]



$\therefore\ \alpha = 4.307.$

(i) From GC, $\alpha = 0.715$ and $\beta = 4.307$.

(ii) $x_{n+1} = \frac{1}{2} e^{\frac{x_n}{2}}$

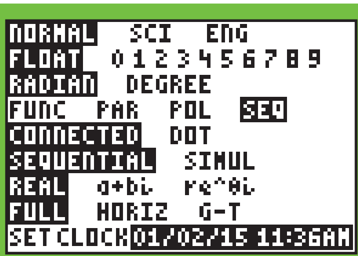
$\Rightarrow 2x_{n+1} = e^{\frac{x_n}{2}}$

$\Rightarrow 2x_{n+1} - e^{\frac{x_n}{2}} = 0$

$\Rightarrow 2L - e^{\frac{L}{2}} = 0$ given that $x_n \to L$ and $x_{n+1} \to L$ when $n \to \infty$.

Since α and β are the roots of $2x - e^{\frac{x}{2}} = 0$, hence x_n converges to α or β if the sequence converges.

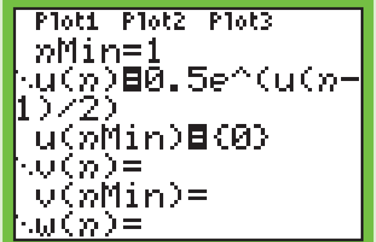
(iii) [MODE] [▼][▼][▼][▶][▶][▶] [ENTER]



Enter **MODE** settings.

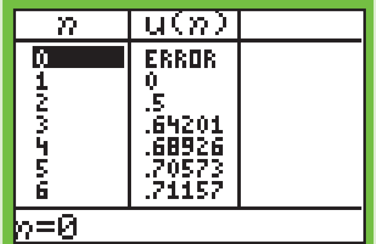
Select **SEQ**.

[Enter the data as shown.]



Set $u_1 = 0$. ☑

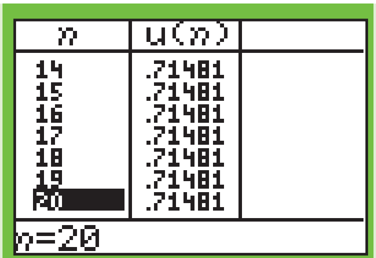
[2nd][GRAPH]



Select **TABLE**.

▶ …

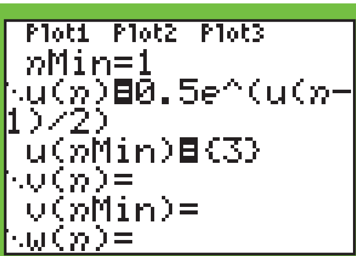
[Observe the change in u_n.]

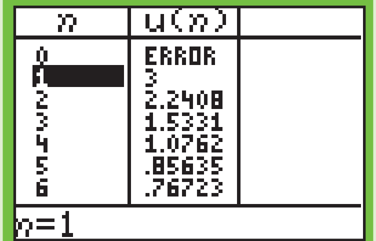


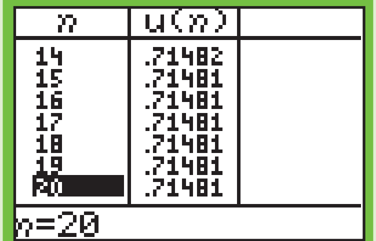
Note that the sequence converges to 0.71481 which is α when $x_1 = 0$. ☑

Repeat the same process for the cases $x_1 = 3$ and $x_1 = 6$.

[Apply similar keystrokes except setting $u_1 = 3$.]



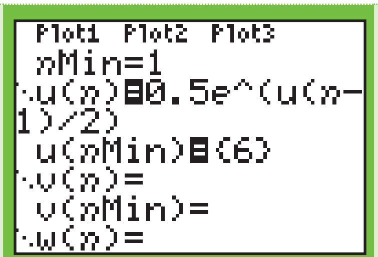


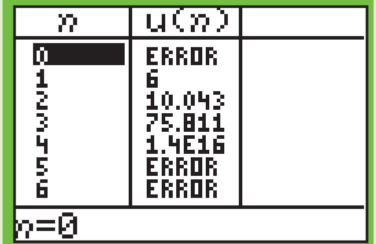


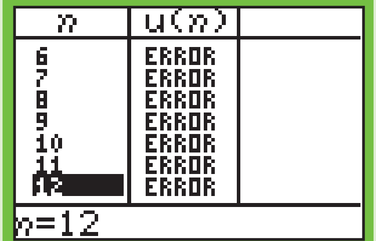
Note that the sequence converges to 0.71481 which is α when $x_1 = 3$. ☑

UNIT 4 SEQUENCES AND SERIES

[Apply similar keystrokes except setting $u_1 = 6$.]







Note that the sequence diverges when $x_1 = 6$ as shown by the ERROR message. ☑

(iv) $x_{n+1} - x_n = \frac{1}{2}e^{\frac{x_n}{2}} - x_n = \frac{1}{2}e^{\frac{x_n}{2}} - x_n = -\frac{1}{2}(2x_n - e^{\frac{x_n}{2}})$

If $\alpha < x_n < \beta$, $2x_n - e^{\frac{x_n}{2}} > 0 \Rightarrow -\frac{1}{2}(2x_n - e^{\frac{x_n}{2}}) < 0 \Rightarrow x_{n+1} - x_n < 0 \Rightarrow x_{n+1} < x_n$.

If $x_n < \alpha$ or $x_n > \beta$, $2x_n - e^{\frac{x_n}{2}} < 0 \Rightarrow -\frac{1}{2}(2x_n - e^{\frac{x_n}{2}}) > 0 \Rightarrow x_{n+1} - x_n > 0 \Rightarrow x_{n+1} > x_n$.

(v) For $x_1 = 0$ where $x_1 < \alpha \Rightarrow x_{n+1} = \frac{1}{2}e^{\frac{x_n}{2}} - x_n < \frac{1}{2}e^{\frac{\alpha}{2}} = \alpha$, hence $x_1 < x_2 < x_3 < ... < \alpha$.

For $x_1 = 3$ where $\alpha < x_1 < \beta \Rightarrow x_n > x_{n+1} = \frac{1}{2}e^{\frac{x_n}{2}} > \frac{1}{2}e^{\frac{\alpha}{2}} = \alpha$, hence $x_1 > x_2 > x_3 > ... > \alpha$.

If $x_1 = 6$ where $x_1 > \beta \Rightarrow x_n < x_{n+1}$, hence $\beta < x_1 < x_2 < x_3 ...$.

GC Techniques covered in this chapter

TECHNIQUES	4.1	4.2	4.3	4.4
Generate a sequence using **seq ( )**.	☑		☑	
Find the **Sum of a Sequence** using **sum ( )** or **cumSum ( )**.	☑		☑	
Find the **Sum to Infinity of a GP**.	☑			
Find the **nth term of a Recurring Sequence** using **SEQ** mode.		☑		
Solve **Quadratic Equation** using **PlySmlt2** application.		☑		
Evaluate Σ.			☑	
Find the **Root(s) of an Equation** under **GRAPH** mode.				☑
Determine the **Behaviour of a Sequence**.				☑